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Global Wellposedness near Maxwellians for Some Kinetic Equations in the Perturbative Framework

By

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Abstract

This survey paper is concerned with the construction of global smooth small-amplitude solutions near Maxwellians to the Cauchy problem of some complex kinetic equations. Some arguments developed recently to yield the corresponding global solvability results in the perturbative framework are reviewed.

§ 1. Kinetic equations

The motion of dilute ionized plasmas consisting of two-species particles (e.g., electrons and ions) under the influence of binary collisions and the self-consistent electromagnetic field can be modelled by the Vlasov-Maxwell-Boltzmann system (called VMB system in the sequel for simplicity) (cf. [5, 37, 41]:

$$\begin{aligned}(1.1)\quad &\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}\left(E+\frac{v}{c}\times B\right)\cdot\nabla_v F_+ = Q(F_+,F_+)+Q(F_+,F_-),\\ &\partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}\left(E+\frac{v}{c}\times B\right)\cdot\nabla_v F_- = Q(F_-,F_+)+Q(F_-,F_-).\end{aligned}$$

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The electromagnetic field $[E, B] = [E(t,x), B(t,x)]$ satisfies the Maxwell equations

$$
\begin{aligned}
\partial_t E - c\nabla_x \times B &= -4\pi \int_{\mathbb{R}^3} v\,(e_+F_+ - e_-F_-)\,dv,\\
\partial_t B + c\nabla_x \times E &= 0,\\
\nabla_x \cdot E &= 4\pi \int_{\mathbb{R}^3} (e_+F_+ - e_-F_-)\,dv,\\
\nabla_x \cdot B &= 0.
\end{aligned}
\tag{1.2}
$$

Here $\nabla_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, $\nabla_v = (\partial_{v_1}, \partial_{v_2}, \partial_{v_3})$. The unknown functions $F_\pm = F_\pm(t,x,v) \geq 0$ are the number density functions for the ions $(+)$ and electrons $(-)$ with position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$, respectively, $e_\pm$ and $m_\pm$ are the magnitudes of charges and masses of the different charged particles, and c is the light speed.

Let $F(v)$, $G(v)$ be two number density functions for two types of particles with masses $m_\pm$ and diameters $\sigma_\pm$, then $Q(F,G)(v)$ is defined as (cf. [4, 5, 22, 37, 41, 58])

$$
\begin{aligned}
Q(F,G)(v) =& \frac{(\sigma_+ + \sigma_-)^2}{4} \int_{\mathbb{R}^3\times\mathbb{S}^2} |u-v|^\gamma \mathbf{b}\left(\frac{\omega\cdot(v-u)}{|u-v|}\right)\{F(v')G(u') - F(v)G(u)\}\,d\omega du\\
\equiv& Q_{\text{gain}}(F,G) - Q_{\text{loss}}(F,G).
\end{aligned}
$$

Here $\omega \in \mathbb{S}^2$ and $\mathbf{b}$, the angular part of the collision kernel, satisfies Grad's cutoff assumption (cf. [25, 58])

$$
0 \leq \mathbf{b}(\cos\theta) \leq C|\cos\theta| \tag{1.3}
$$

for some positive constant $C > 0$. The deviation angle $\pi - 2\theta$ satisfies $\cos\theta = \omega\cdot(v-u)/|v-u|$. Moreover, for $m_1, m_2 \in \{m_+, m_-\}$,

$$
v' = v - \frac{2m_2}{m_1+m_2}[(v-u)\cdot\omega]\omega, \quad u' = u + \frac{2m_1}{m_1+m_2}[(v-u)\cdot\omega]\omega,
$$

which denote velocities (v', u') after a collision of particles having velocities (v, u) before the collision and vice versa. Notice that the above identities follow from the conservation of momentum $m_1 v + m_2 u$ and energy $\frac{1}{2}m_1|v|^2 + \frac{1}{2}m_2|u|^2$.

The exponent $\gamma \in (-3, 1]$ in the kinetic part of the collision kernel is determined by the potential of intermolecular force, which is classified into the soft potential case when $-3 < \gamma < 0$, the Maxwell molecular case when $\gamma = 0$, and the hard potential case when $0 < \gamma \leq 1$ which includes the hard sphere model with $\gamma = 1$ and $\mathbf{b}(\cos\theta) = C|\cos\theta|$ for some positive constant $C > 0$. For the soft potentials, the case $-2 \leq \gamma < 0$ is called the moderately soft potentials while $-3 < \gamma < -2$ is called the very soft potentials, cf. [60] by Villani. The importance and the difficulty in studying the very soft potentials can be also found in that review paper.

In physical situations the ion mass is usually much larger than the electron mass so that the electrons move much faster than the ions. Thus, the ions are often described by a fixed ion background and only the electrons move. For such a simple case, after some normalization, the two-species VMB system (1.1)-(1.2) is then reduced to the following one-species VMB system:

$$\begin{aligned}
&\partial_t F + v\cdot\nabla_x F + (E + v\times B)\cdot\nabla_v F = Q(F,F),\\
&\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} vF dv,\\
&\partial_t B + \nabla_x\times E = 0,\\
&\nabla_x\cdot E = \int_{\mathbb{R}^3} F dv - n_b(x),\\
&\nabla_x\cdot B = 0.
\end{aligned}\tag{1.4}$$

Here $F(t,x,v)\geq 0$ denotes the number density function for the electrons at time $t\geq 0$ with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$, $E(t,x)$ and $B(t,x)$ denote the electric and magnetic fields, respectively. $n_b(x)$ is the density of the ionic background and is assumed throughout this paper to be a positive constant which means that it is spatially uniform.

A somewhat simpler model, which can be derived formally by letting the light speed $c\to+\infty$ in (1.1)-(1.2), is the so-called two-species Vlasov-Poisson-Boltzmann system (called VPB system in the sequel for simplicity), where $B(t,x)\equiv 0$ and $E(t,x)\equiv \nabla_x\phi(t,x)$ in (1.1)-(1.2), and $\phi(t,x)$, the potential of the electric field, solves the Poisson euqation:

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}\nabla_x\phi\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_+,F_-),\\
&\partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}\nabla_x\phi\cdot\nabla_v F_- = Q(F_-,F_+) + Q(F_-,F_-),\\
&\Delta\phi = 4\pi\int_{\mathbb{R}^3}\left(e_+F_+ - e_-F_-\right)dv,\\
&\lim_{|x|\to+\infty}\phi(t,x) = 0,
\end{aligned}\tag{1.5}$$

where $F_\pm = F_\pm(t,x,v)\geq 0$ denotes the number density functions for the ions $(+)$ and electrons $(-)$, respectively, at time $t\geq 0$ with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$.

Similarly, one can derive the following one-species VPB system from (1.4)

$$\begin{aligned}
&\partial_t F + v\cdot\nabla_x F + \nabla_x\phi\cdot\nabla_v F = Q(F,F),\\
&\Delta\phi = \int_{\mathbb{R}^3} F dv - n_b(x),\\
&\lim_{|x|\to+\infty}\phi(t,x) = 0.
\end{aligned}\tag{1.6}$$

Here $F(t,x,v)\geq 0$ stands for the number density function for the electrons at time

$t \geq 0$, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$ and $\phi(t,x)$ represents for the potential of the electric field.

Notice that (cf. [32, 60]) the Coulomb potential coincides with the limit at $\gamma = -3$ for which the Boltzmann collision operator $Q(F,G)(v)$ should be replaced by the Landau operator $\mathcal{Q}(F,G)(v)$ under the grazing collision:

$$\begin{aligned}
\mathcal{Q}\left(G_\pm, G_\mp\right) &= \frac{c_{\pm\mp}}{m_\pm}\nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v-v')\left\{\frac{G_\pm(v')\nabla_v G_\mp(v)}{m_\mp} - \frac{\nabla_{v'} G_\pm(v')G_\mp(v)}{m_\pm}\right\} dv' \\
&= \frac{c_{\pm\mp}}{m_\pm}\sum_{i,j=1}^{3}\partial_i \int_{\mathbb{R}^3} \Phi^{ij}(v-v')\left\{\frac{G_\pm(v')\partial_j G_\mp(v)}{m_\mp} - \frac{\partial_j G_\pm(v')G_\mp(v)}{m_\pm}\right\} dv',
\end{aligned}$$

where $\partial_i = \partial_{v_i}$ and $\Phi(v) = \left(\Phi^{ij}(v)\right)_{3\times 3}$ is the famous Landau (Fokker-Planck) kernel:

$$\Phi^{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)|v|^{\gamma+2}, \quad \gamma \geq -3 \tag{1.7}$$

and $\gamma = -3$ corresponds to the Coulomb potential. Here c is the light speed, $e_\pm$ and $m_\pm$ are the magnitude of charges and masses for different charged particles, $c_{\pm\mp} = 2\pi e_\pm^2 e_\mp^2 \ln\Lambda$, $\ln\Lambda = \ln\left(\frac{\lambda_D}{b_0}\right)$, $\lambda_D = \left(\frac{T_0}{4\pi n_e e^2}\right)^{\frac{1}{2}}$ is the Debye shielding distance and $b_0 = \frac{e^2}{3T_0}$ is the typical distance of closest approach' for a thermal particle. Note that the $\pm$ and the $\mp$ signify the possibility of either the $+$ or the $-$ in the sign configuration.

If the Boltzmann collision operator $Q(F,G)$ in (1.1), (1.4), (1.5) and (1.6) is replaced by the Landau collision operator $\mathcal{Q}(F,G)$, then one can get the following two-species Vlasov-Maxwell-Landau system (called VML system in the sequel for simplicity):

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}\left(E + \frac{v}{c}\times B\right)\cdot\nabla_v F_+ = \mathcal{Q}(F_+,F_+) + \mathcal{Q}(F_+,F_-),\\
&\partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}\left(E + \frac{v}{c}\times B\right)\cdot\nabla_v F_- = \mathcal{Q}(F_-,F_+) + \mathcal{Q}(F_-,F_-),\\
&\partial_t E - c\nabla_x \times B = -4\pi\int_{\mathbb{R}^3} v\left(e_+F_+ - e_-F_-\right)dv,\\
&\nabla_x\cdot E = 4\pi\int_{\mathbb{R}^3}\left(e_+F_+ - e_-F_-\right)dv,\\
&\partial_t B + c\nabla_x\times E = 0, \quad \nabla_x\cdot B = 0,
\end{aligned} \tag{1.8}$$

the one-species VML system:

$$\begin{aligned}
&\partial_t F + v\cdot\nabla_x F + (E + v\times B)\cdot\nabla_v F = \mathcal{Q}(F,F),\\
&\partial_t E - \nabla_x\times B = -\int_{\mathbb{R}^3} vF dv,\\
&\partial_t B + \nabla_x\times E = 0,\\
&\nabla_x\cdot E = \int_{\mathbb{R}^3} F dv - n_b(x),\\
&\nabla_x\cdot B = 0,
\end{aligned} \tag{1.9}$$

the two-species Vlasov-Poisson-Landau system (called VPL system in the sequel for simplicity):

$$
\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ - \frac{e_+}{m_+}\nabla_x\phi\cdot\nabla_v F_+ = \mathcal{Q}(F_+,F_+)+\mathcal{Q}(F_+,F_-),\\
&\partial_t F_- + v\cdot\nabla_x F_- + \frac{e_-}{m_-}\nabla_x\phi\cdot\nabla_\xi F_- = \mathcal{Q}(F_-,F_+)+\mathcal{Q}(F_-,F_-),\\
&-\triangle_x\phi = 4\pi\rho = 4\pi\int_{\mathbb{R}^3}(e_+F_+ - e_-F_-)\,dv,\\
&\lim_{|x|\to+\infty}\phi(t,x)=0,
\end{aligned}
\tag{1.10}
$$

and the one-species VPL system:

$$
\begin{aligned}
&\partial_t F + v\cdot\nabla_x F + \nabla_x\phi\cdot\nabla_v F = \mathcal{Q}(F,F),\\
&\triangle_x\phi = \int_{\mathbb{R}^3} F\,dv - n_b(x),\\
&\lim_{|x|\to+\infty}\phi(t,x)=0,
\end{aligned}
\tag{1.11}
$$

respectively. Here $F_\pm(t,x,v)\geq 0$ denote the number density functions for the ions (+) and electrons (−) respectively, at time $t\geq 0$, position $x\in\mathbb{R}^3$, velocity $v\in\mathbb{R}^3$, while $F(t,x,v)\geq 0$ stands for the number density function for the electrons at time $t\geq 0$, position $x\in\mathbb{R}^3$, velocity $v\in\mathbb{R}^3$.

§ 2. The problem

It is easy to see that equilibrium states of the two-species VMB system (1.1)-(1.2), the two-species VML system (1.8), the two-species VPB system (1.5) and the two-species VPL system (1.10) system are

$$
\begin{aligned}
\mu_+(v) &= \frac{n_0}{e_+}\left(\frac{m_+}{2\pi\kappa_B T_0}\right)^{\frac{3}{2}}\exp\left(-\frac{m_+|v|^2}{2\kappa_B T_0}\right),\\
\mu_-(v) &= \frac{n_0}{e_-}\left(\frac{m_-}{2\pi\kappa_B T_0}\right)^{\frac{3}{2}}\exp\left(-\frac{m_-|v|^2}{2\kappa_B T_0}\right)
\end{aligned}
$$

with $E(t,x)=B(t,x)\equiv 0$ or $\phi(t,x)\equiv 0$, where where $\kappa_B>0$ is the Boltzmann constant, $n_0>0$ and $T_0>0$ are constant reference number density and temperature, respectively. While for the corresponding one-species models (1.4), (1.6), (1.9) and (1.11), any global Maxwellians together with $E(t,x)=B(t,x)\equiv 0$ or $\phi(t,x)\equiv 0$ are their equilibrium states.

Since the presence of all the physical constants and all other involving constants such as the generic constant 4π, etc., does not create essential mathematical difficulties and due to the fact that our analysis does not use the cancelation phenomenon between different types of particles (cf. [64, 63, 62]), thus without loss of generality, we can normalize all these constants in the two-species VMB system (1.1)- (1.2), the two-species VML system (1.8), the two-species VPB system (1.5) and the two-species VPL system (1.10) system to be one. Accordingly, $\mu_\pm(v)$ are normalized as $\mu = \mu_-(v) = \mu_+(v) = (2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}$.

The problem to be considered is on the construction of global smooth solutions to the Cauchy problem of the above eight systems near the above normalized equilibrium states with prescribed initial condition for the whole range of the parameter γ under consideration. In the following, we illustrate our main concerns by only considering the two-species VMB system (1.1)-(1.2) and the one-species VPL system (1.6) with the following initial conditions:

- Initial condition for the two-species VMB system (1.1)-(1.2):

$$F_\pm(0,x,v) = F_{0,\pm}(x,v),\ E(0,x) = E_0(x),\ B(0,x) = B_0(x), \tag{2.1}$$

which satisfy the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} (F_{0,+} - F_{0,-})dv, \quad \nabla_x \cdot B_0 = 0; \tag{2.2}$$

- Initial condition for the one-species VPL system (1.11):

$$F(0,x,v) = F_0(x,v). \tag{2.3}$$

§ 3. Reformulations of the problems

§ 3.1. The two-species VMB system

For the two-species VMB system (1.1)-(1.2), if we define the perturbation $f_\pm = f_\pm(t,x,v)$ by

$$F_\pm(t,x,v) = \mu + \mu^{1/2} f_\pm(t,x,v),$$

then, the Cauchy problem (1.1)-(1.2), (2.1) is reformulated as

$$\begin{aligned}
&\partial_t f_\pm + v\cdot\nabla_x f_\pm \pm (E + v\times B)\cdot\nabla_v f_\pm \mp E\cdot v\mu^{1/2} \mp \frac{1}{2}E\cdot v f_\pm + L_\pm f\\
&\qquad\qquad = \Gamma_\pm(f,f),\\
&\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} v\mu^{1/2}(f_+ - f_-)dv,\\
&\partial_t B + \nabla_x \times E = 0,\\
&\qquad \nabla_x \cdot E = \int_{\mathbb{R}^3} \mu^{1/2}(f_+ - f_-)dv, \quad \nabla_x\cdot B = 0
\end{aligned} \tag{3.1}$$

with initial data

$$f_\pm(0,x,v) = f_{0,\pm}(x,v), \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \tag{3.2}$$

which satisfy the compatibility conditions

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3} \mu^{1/2}(f_{0,+} - f_{0,-})dv, \quad \nabla_x \cdot B_0 = 0. \tag{3.3}$$

Set

$$f = [f_+, f_-],$$

then $(3.1)_1$ can be written as

$$\partial_t f + v\cdot\nabla_x f + q_0(E + v\times B)\cdot\nabla_v f - E\cdot v\mu^{1/2}q_1 + Lf = \frac{q_0}{2}E\cdot vf + \Gamma(f,f). \tag{3.4}$$

Here $q_0 = \mathrm{diag}(1,-1)$, $q_1 = [1,-1]$, the linearized collision operator Lf and the nonlinear collision term $\Gamma(f,f)$ are respectively defined by

$$Lf = [L_+f, L_-f], \qquad \Gamma(f,g) = [\Gamma_+(f,g), \Gamma_-(f,g)] \tag{3.5}$$

with

$$\begin{aligned} L_\pm f &= -\mu^{-1/2}\left\{Q\left(\mu, \mu^{1/2}(f_\pm + f_\mp)\right) + 2Q\left(\mu^{1/2}f_\pm, \mu\right)\right\},\\ \Gamma_\pm(f,g) &= \mu^{-1/2}\left\{Q\left(\mu^{1/2}f_\pm, \mu^{1/2}g_\pm\right) + Q\left(\mu^{1/2}f_\pm, \mu^{1/2}g_\mp\right)\right\}. \end{aligned}$$

For the linearized Boltzmann collision operator L, it is easy to see that (cf. [25, 33])

- L is non-negative and the null space $\mathcal{N}$ of L is given by

$$\mathcal{N} = \mathrm{span}\left\{[1,0]\mu^{1/2}, [0,1]\mu^{1/2}, [v_i,v_i]\mu^{1/2}(1\le i\le 3), [|v|^2,|v|^2]\mu^{1/2}\right\}.$$

- Under Grad's angular cutoff assumption (1.3), L can be decomposed as

$$Lf = \nu f - Kf \tag{3.6}$$

with the collision frequency $\nu(v)$ and the nonlocal integral operator $K = [K_+, K_-]$ being defined by

$$\nu(v) = 2Q_{\mathrm{loss}}(1,\mu) = 2\int_{\mathbb{R}^3\times\mathbb{S}^2}|v-u|^\gamma \mathbf{b}\left(\frac{\omega\cdot(v-u)}{|v-u|}\right)\mu(u)d\omega du \sim (1+|v|)^\gamma \tag{3.7}$$

and

$$\begin{aligned} (K_\pm f)(v) &= \mu^{-\frac12}\left\{2Q_{\mathrm{gain}}\left(\mu^{\frac12}f_\pm, \mu\right) - Q\left(\mu, \mu^{\frac12}(f_\pm + f_\mp)\right)\right\}\\ &= \int_{\mathbb{R}^3\times\mathbb{S}^2}|u-v|^\gamma\mathbf{b}\left(\frac{\omega\cdot(v-u)}{|v-u|}\right)\mu^{\frac12}(u)\left\{2\mu^{\frac12}(u')f_\pm(v')\right.\\ &\qquad \left.-\mu^{\frac12}(v')(f_\pm + f_\mp)(u') + \mu^{\frac12}(v)(f_\pm + f_\mp)(u)\right\}d\omega du. \end{aligned} \tag{3.8}$$

Here and in the rest of this paper $A \sim B$ means that there exists some generic positive constant $C > 1$ such that $C^{-1}B \leq A \leq CB$.

Define $\mathbf{P}$ as the orthogonal projection from $L^2(\mathbb{R}^3_v) \times L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$, then for any given function $f(t,x,v) \in L^2(\mathbb{R}^3_v)$, one has

$$\begin{aligned}\mathbf{P}f = & \, a_+(t,x)[1,0]\mu^{1/2} + a_-(t,x)[0,1]\mu^{1/2} \\ & + \sum_{i=1}^3 b_i(t,x)[1,1]v_i\mu^{1/2} \\ & + c(t,x)[1,1](|v|^2-3)\mu^{1/2}\end{aligned}$$

with

$$\begin{aligned}a_\pm &= \int_{\mathbb{R}^3} \mu^{1/2} f_\pm dv, \\ b_i &= \frac{1}{2}\int_{\mathbb{R}^3} v_i \mu^{1/2}(f_+ + f_-)dv, \quad i = 1,2,3, \\ c &= \frac{1}{6}\int_{\mathbb{R}^3} (|v|^2-3)\mu^{1/2}(f_+ + f_-)dv.\end{aligned}$$

As in [30, 33], one can define the following macro-micro decomposition of the solution of the two-species VMB system (3.4) with respect to a given global Maxwellian:

$$f(t,x,v) = \underbrace{\mathbf{P}f(t,x,v)}_{\text{fluid component}} + \underbrace{\{\mathbf{I}-\mathbf{P}\}f(t,x,v)}_{\text{non-fluid component}}. \tag{3.9}$$

Under Grad's angular cutoff assumption (1.3), L is locally coercive in the sense that, cf. [4, 22, 29, 33]

$$-\langle f, Lf\rangle \geq \sigma_0 \left|\{\mathbf{I}-\mathbf{P}\}f\right|_\nu^2 \equiv \sigma_0 \left\|\sqrt{\nu}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{L^2(\mathbb{R}^3_v)}, \quad \nu(v) \sim (1+|v|)^\gamma \tag{3.10}$$

holds for some positive constant $\sigma_0 > 0$.

To deduce the macroscopic equations, we define the high-order moment functions $\Theta(f_\pm) = (\Theta_{ij}(f_\pm))_{3\times 3}$ and $\Lambda(f_\pm) = (\Lambda_1(f_\pm), \Lambda_2(f_\pm), \Lambda_3(f_\pm))$ by

$$\begin{aligned}\Theta_{ij}(f_\pm) &= \left\langle (v_iv_j - 1)\mu^{1/2}, f_\pm \right\rangle, \\ \Lambda_i(f_\pm) &= \frac{1}{10}\left\langle (|v|^2-5)v_i\mu^{1/2}, f_\pm\right\rangle,\end{aligned}$$

one has by taking velocity integrations of (3.4) with respect to the velocity moments $\mu^{1/2}$, $v_i\mu^{1/2}$ $(i = 1,2,3)$, $\frac{1}{6}(|v|^2-3)\mu^{1/2}$ that

$$
\begin{aligned}
&\partial_t a_\pm + \nabla_x\cdot b + \nabla_x\cdot\left\langle v\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle = \left\langle \mu^{1/2}, S_\pm\right\rangle,\\
&\partial_t\left[b_i + \left\langle v_i\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle\right] + \partial_{x_i}(a_\pm+2c)\mp E_i\\
&\qquad\qquad +\nabla_x\cdot\left\langle vv_i\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle = \left\langle v_i\mu^{1/2}, S_\pm - L_\pm f\right\rangle,\\
&\partial_t\left[c+\frac{1}{6}\left\langle (|v|^2-3)\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle\right] + \frac{1}{3}\nabla_x\cdot b\\
&\qquad\qquad\quad +\frac{1}{6}\nabla_x\cdot\left\langle \left(|v|^2-3\right)v\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle\\
&\qquad\qquad = \frac{1}{6}\left\langle (|v|^2-3)\mu^{1/2}, S_\pm - L_\pm f\right\rangle.
\end{aligned}
\tag{3.11}
$$

Further taking velocity integrations of (3.4) with respect to the above high-order moments one has

$$
\begin{aligned}
&\partial_t\left[\Theta_{ii}(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f)+2c\right] + 2\partial_{x_i}b_i = \Theta_{ii}(r_\pm+S_\pm),\\
&\partial_t\Theta_{ij}(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f) + \partial_{x_j}b_i + \partial_{x_i}b_j + \nabla_x\cdot\left\langle v\mu^{1/2}, \{\mathbf{I}_\pm-\mathbf{P}_\pm\}f\right\rangle\\
&\qquad\qquad\qquad\qquad = \Theta_{ij}(r_\pm+S_\pm) + \left\langle \mu^{1/2}, S_\pm\right\rangle, \quad i\neq j,\\
&\partial_t\Lambda_i(\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f) + \partial_{x_i}c = \Lambda_i(r_\pm+S_\pm),
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
r_\pm &= -v\cdot\nabla_x\{\mathbf{I}_\pm-\mathbf{P}_\pm\}f - L_\pm f,\\
S\equiv[S_-,S_+] &= \Gamma(f,f)+\frac{1}{2}q_0E\cdot vf - q_0(E+v\times B)\cdot\nabla_v f.
\end{aligned}
$$

§ 3.2. The one-species VPL system

For the one-species VPL system (1.11), if we let

$$
F(t,x,v) = \mu + \mu^{\frac{1}{2}}u(t,x,v),
$$

then the Cauchy problem (1.11)-(2.3) can be rewritten as

$$
\begin{aligned}
&\partial_t u + v\cdot\nabla_x u + \nabla_x\phi\cdot\nabla_v u - \frac{1}{2}v\cdot\nabla_x\phi u - \nabla_x\phi\cdot v\mu^{\frac{1}{2}} + \mathcal{L}u = \mathbf{\Gamma}(u,u),\\
&\triangle_x\phi = \int_{\mathbb{R}^3}\mu^{\frac{1}{2}}u\,dv, \qquad \lim_{|x|\to+\infty}\phi(t,x) = 0,
\end{aligned}
\tag{3.13}
$$

$$
u(0,x,v) = u_0(x,v). \tag{3.14}
$$

Here the linearized Landau collision operator $\mathcal{L}$ and the nonlinear Landau collision term $\Gamma(u,u)$ are given by

$$\begin{aligned}\mathcal{L}u &= -\mu^{-1/2}\left\{\mathcal{Q}\left(\mu,\mu^{\frac12}u\right)+\mathcal{Q}\left(\mu^{\frac12}u,\mu\right)\right\},\\ \mathbf{\Gamma}(u,u) &= \mu^{-\frac12}\mathcal{Q}\left(\mu^{\frac12}u,\mu^{\frac12}u\right),\end{aligned}\tag{3.15}$$

respectively.

It is easy to see that the null space $\mathcal{N}$ of the linearized Landau collision operator $\mathcal{L}$ is (cf. [28, 32])

$$\mathcal{N}=\operatorname{Span}\left\{\nu^{\frac12},\ v_i\mu^{\frac12}(1\le i\le 3),\ \left(|v|^2-3\right)\mu^{\frac12}\right\}.$$

Similarly, if we define $\mathbf{P}$ as the orthogonal projection from $L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$, then for any given function $u(t,x,v)\in L^2(\mathbb{R}^3_v)$, one has

$$\mathbf{P}u=a(t,x)\mu^{\frac12}+b(t,x)\cdot v\mu^{\frac12}+c(t,x)(|v|^2-3)\mu^{\frac12},\quad \mathbf{P_0}u=a(t,x)\mu^{\frac12}$$

with

$$\begin{aligned}a &= \int_{\mathbb{R}^3}\mu^{\frac12}u\,dv=\int_{\mathbb{R}^3}\mu^{\frac12}\mathbf{P}udv,\\ b_i &= \int_{\mathbb{R}^3}v_i\mu^{1/2}u\,dv=\int_{\mathbb{R}^3}v_i\mu^{\frac12}\mathbf{P}u\,dv,\quad i=1,2,3,\\ c &= \frac16\int_{\mathbb{R}^3}(|v|^2-3)\mu^{\frac12}udv=\frac16\int_{\mathbb{R}^3}(|v|^2-3)\mu^{\frac12}\mathbf{P}udv.\end{aligned}$$

Thus one has the following macro-microscopic decomposition

$$u(t,x,v)=\underbrace{\mathbf{P}u(t,x,v)}_{u_1:\ \text{fluid part}}+\underbrace{\{\mathbf{I}-\mathbf{P}\}u(t,x,v)}_{u_2:\ \text{nonfluid part}}\tag{3.16}$$

and the linearized Landau collision operator $\mathcal{L}$ is locally coercive in the sense that, cf. [6, 28, 32, 48]

$$\begin{aligned}&\langle\mathcal{L}f,f\rangle\gtrsim|\{\mathbf{I}-\mathbf{P}\}f|^2_\sigma,\\ &|f|_\sigma\equiv\left\|\langle v\rangle^{\frac{\gamma+2}{2}}f\right\|_{L^2_v}+\underbrace{\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_vf\cdot\frac{v}{|v|}\right\|_{L^2_v}+\left\|\langle v\rangle^{\frac{\gamma+2}{2}}\nabla_vf\times\frac{v}{|v|}\right\|_{L^2_v}}_{\gtrsim\left\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_vf\right\|_{L^2_v}}.\end{aligned}\tag{3.17}$$

For any $g=g(v)$, define moment functions $\Theta_{ij}(g)$ and $\Lambda_i(g),\ 1\le i,j\le 3$, by

$$\Theta_{ij}(g)=\int_{\mathbb{R}^3}(v_iv_j-1)\mu^{1/2}vdv,\quad \Lambda_i(g)=\frac{1}{10}\int_{\mathbb{R}^3}(|v|^2-5)v_i\mu^{1/2}vdv,$$

then, one can derive form (3.13) a fluid-type system of equations

$$\begin{aligned} \partial_t a+\nabla_x\cdot b&=0,\\ \partial_t b+\nabla_x\cdot(a+2c)+\nabla_x\cdot\Theta(u_2)-\nabla_x\phi&=\nabla_x\phi a,\\ \partial_t c+\frac{1}{3}\nabla_x\cdot b+\frac{5}{3}\nabla_x\cdot\Lambda(u_2)&=\frac{1}{3}\nabla_x\phi\cdot b,\\ \triangle_x\phi&=a \end{aligned}\tag{3.18}$$

and

$$\begin{aligned} \partial_t\Theta_{ij}(u_2)+\partial_{x_i}b_j+\partial_{x_j}b_i-\frac{2}{3}\delta_{ij}\nabla_x\cdot\Lambda(u_2)&=\Theta_{ij}(r+G)-\frac{2}{3}\delta_{ij}\nabla_x\phi\cdot b,\\ \partial_t\Lambda_i(u_2)+\partial_{x_i}c&=\Lambda_i(r+G) \end{aligned}\tag{3.19}$$

with

$$r=-\xi\cdot\nabla_x u_2-\mathcal{L}u_2,\quad G=\mathbf{\Gamma}(u,u)+\frac{1}{2}v\cdot\nabla_x\phi u-\nabla_x\phi\cdot\nabla_v u,\tag{3.20}$$

where r is a linear term related only to the microscopic component u_2 and G is a quadratic nonlinear term.

§ 4. Method and main difficulties

The main method to deduce the global solvability results to the Cauchy problem of the above complex kinetic equations is the nonlinear energy method developed in [30, 49, 50]) for the Boltzmann equation, which is based on the macro-microscopic decomposition on their solutions (cf. (3.9) and (3.16)) and on the equations themselves (cf. (3.11)-(3.12) and (3.18)-(3.19) for the corresponding fluid equations for two-species VMB system and one-species VPL system, respectively). The dissipative mechanism used to control the fluid component is the compressible Navier-Stokes type dissipation, while for non-fluid component, it is the microscopic $H-$theorems (3.10) and (3.17) and their weighted versions.

If the external forces are treated as perturbative terms, some satisfactory global solvability results can be established for the Boltzmann type equations for hard sphere model (cf. [15] for the Boltzmann equation with given potential force, [13, 16, 31, 34, 63, 68, 70, 71] for the VPB system and [7, 14, 33, 40, 53] for the VMB system) and the Landau type equations for the case of $\gamma\geq -1$. In such a case, it is unnecessary to use the time-velocity weighted energy method.

But to yield a satisfactory wellposedness theory for the above system for the whole range of the parameter γ under consideration, that is, $\gamma>-3$ for the Boltzmann type eqautions and $\gamma\geq -3$ for the Landau type equatiuons, since the fluid component can be estimated similar to that of the Boltzmann type equations for hard sphere model or the

Landau type equations for $\gamma \geq -1$, the problem is then reduced to control the possible growth of the non-fluid component suitably which can be summarized as follows:

(i). How to control the possible growth of the non-fluid component induced by the following two types of nonlinear terms: the nonlinear collision term and the nonlinear terms caused by the Lorenz force like $(E+v\times B)\cdot\nabla_v f$, $E\cdot vf$ for VMB system (3.4) and $\nabla_x\phi\cdot\nabla_v u$, $v\cdot\nabla_x\phi u$ for VPL system (3.13), especially the difficulty caused by the degeneracy of the microscopic $H-$theorems (3.10) and (3.17) for large velocity and the velocity growth of the first order in these terms;

(ii). If the weighted energy method is needed, do similar coercive estimates in the weighted version hold for the linearized collision operators L and $\mathcal{L}$? how to deal with the linear transport term $v\cdot\nabla_x f$ for VMB system (3.4) and $v\cdot\nabla_x u$ for VPL system (3.13)?

(iii). How to deal with the regularity-loss of the electromagnetic field $[E(t,x),B(t,x)]$ for Maxwell equations?

Since the coercive estimates in the weighted version for the linearized Boltzmann collision operator L and the linearized Landau collision operator $\mathcal{L}$ and the weighted estimates on the nonlinear Boltzmann collision operator $\Gamma(f,f)$ and the nonlinear Landau collision operator $\mathbf{\Gamma}(f,f)$ are well-established in [32, 62] for the Landau collision operators $\mathcal{L}$ and $\mathbf{\Gamma}(f,f)$ defined by (3.15) and in [18, 55] for the Boltzmann collision operators L and $\Gamma(f,f)$ defined by (3.5), the main difficulties we encountered are the following:

- How to control the possible growth of the non-fluid component induced by the external forces such as the Lorenz force like $(E+v\times B)\cdot\nabla_v f$, $E\cdot vf$ for VMB system (3.4) and $\nabla_x\phi\cdot\nabla_v u$, $v\cdot\nabla_x\phi u$ for VPL system (3.13)?
- How to deal with the linear transport term $v\cdot\nabla_x f$ for VMB system (3.4) and $v\cdot\nabla_x u$ for VPL system (3.13) if time-velocity weighted energy method is used?
- How to deal with the regularity-loss of the electromagnetic field $[E(t,x),B(t,x)]$ for Maxwell equations?

§ 5. An argument of Y. Guo

An breakthrough to overcome the above mentioned difficulties was made by Y. Guo in [32] when dealing with the problem on the global solvability of the Cauchy problem of the two-species VPL system (1.10) near Maxwellians for the Coulomb potential. The main ideas of Y. Guo in [32] can be outlined as follows:

- A new velocity weight function

$$\overline{w}_{l-|\alpha|-|\beta|}(v) = \langle v\rangle^{-(\gamma+1)(l-|\alpha|-|\beta|)}, \quad \langle v\rangle = \sqrt{1+|v|^2}, \quad l \geq |\alpha| + |\beta|$$

  is introduced to capture the weak velocity diffusion in the linearized Landau kernel $\mathcal{L}$ for the case of $-3 \leq \gamma < -2$;

- An exponential weight of electric potential $e^{-\phi}$ is used to cancel the growth of the velocity in the nonlinear term $-\frac{1}{2}v \cdot \nabla_x \phi \partial^\alpha_\beta u_2$ induced by the electric force:

$$\begin{aligned}
&e^{-\phi}\overline{w}^2_{\ell-|\alpha|-|\beta|}(v)\partial^\alpha_\beta u_2 \left(-\frac{1}{2}v \cdot \nabla_x\phi\partial^\alpha_\beta u_2 + v \cdot \nabla_x \partial^\alpha_\beta u_2\right)\\
=&\frac{1}{2}v \cdot \nabla_x \left(e^{-\phi}\overline{w}^2_{\ell-|\alpha|-|\beta|}(v)\left|\partial^\alpha_\beta u_2\right|^2\right).
\end{aligned}$$

  Here the fact that the electric field is a potential force plays an essential role;

- A decay of the electric field $\phi(t,x)$ is employed to close the energy estimate, which asks that $\|\partial_t\phi(t)\|_{L^\infty} \in L^1(\mathbb{R}^+)$.

We note, however, that, on the one hand, for the one-species VPL system (3.13), even if the initial perturbation is assumed to satisfy the neutral condition

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}}(v)u_0(x,v)dvdx = 0, \tag{5.1}$$

one can only get that (cf. [69] for the spectrum analysis for some kinetic equations including the VPL system)

$$\|\partial_t\phi(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim (1+t)^{-1}, \tag{5.2}$$

on the other hand, for the VML systems (1.9) and (1.8), the external forces involved are the Lorenz forces which are no longer potential forces, thus it would be interesting to see how to generalize the arguments developed in [32] to deal with the problem on the construction of global smooth small-amplitude solutions to the one species VPL system (3.14) and the VML systems (1.9) and (1.8).

Moreover, for the Boltzmann type equations, the dissipative of the linearized Boltzmann collision operator L is weaker than that of the linearized Landau collision operator $\mathcal{L}$, then the problem is how to obtain the corresponding global wellposedness theories in the perturbative framework for the VPB systems (1.4) and (1.5) and the VMB system (1.4) and (1.1)?

In the rest of this paper, we will outline our three arguments to deal with the above two problems.

§ 6. Our first argument

Let's explain our first argument by using the two-species VMB system (3.4). *One of our main ideas is to introduce the following new weight function* $w_{\ell-|\beta|,\kappa}(t,v)$:

$$w_{\ell-|\beta|,\kappa}(t,v)=\langle v\rangle^{\kappa(\ell-|\beta|)}\exp\left(\frac{q\langle v\rangle^2}{(1+t)^{\vartheta}}\right),\quad 0<q\ll 1, \tag{6.1}$$

which results in, in addition to the dissipative term which is due to the weighted version of the coercive estimate of L

$$D^L_{\ell-|\beta|,\kappa}\equiv\left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu, \tag{6.2}$$

the extra dissipative term

$$D^W_{\ell-|\beta|,\kappa}\equiv(1+t)^{-1-\vartheta}\left\|\langle v\rangle w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \tag{6.3}$$

induced by the exponent factor $\exp\left(\frac{q\langle v\rangle^2}{(1+t)^{\vartheta}}\right)$ of the weight function $w_{\ell-|\beta|,\kappa}(t,v)$.

Now the main difficulties mentioned in Section 4 to control the possible growth of the non-fluid component $\{\mathbf{I}-\mathbf{P}\}f$ are:

- How to control the nonlinear terms containing $[E(t,x),B(t,x)]$ suitably? Especially

$$I^E_{\ell-|\beta|,\kappa}\equiv\sum_{|\alpha_1|\geq 1}\left(\partial^{\alpha_1}E\cdot\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f,w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right) \tag{6.4}$$

  and

$$I^B_{\ell-|\beta|,\kappa}\equiv\sum_{|\alpha_1|\geq 1}\left((v\times\partial^{\alpha_1}B)\cdot\nabla_v\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f,w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right); \tag{6.5}$$

- How to deal with the term

$$I^{lt}_{\ell-|\beta|,\kappa}\equiv\left(\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,w^2_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right) \tag{6.6}$$

  related to the linear transport term $v\cdot\nabla_x f$?

- How to deal with the regularity-loss of the electromagnetic field $[E(t,x),B(t,x)]$ for Maxwell equations?

Our second idea is to control the linear transport term suitably first and then try to close the energy estimate through one set of time-velocity weighted energy estimate with respect to the weight function $w_{\ell-|\beta|,\kappa}(t,\xi)$. For this purpose, we had to choose $\kappa=-\gamma$ and due to $w^2_{\ell-|\beta|,-\gamma}=w_{\ell-|\beta|,-\gamma}\times w_{\ell-|\beta-e_i|,-\gamma}\times\langle v\rangle^{\gamma}$, the term $I^{lt}_{\ell-|\beta|,-\gamma}$ can be controlled by $D^L_{\ell-|\beta|,-\gamma}$, while since $w^2_{\ell-|\beta|,-\gamma}=w_{\ell-|\beta|,-\gamma}\times w_{\ell-|\beta+e_i|,-\gamma}\times\langle v\rangle^{-\gamma}$, to control $I^B_{\ell-|\beta|,-\gamma}$ by $D^W_{\ell-|\beta|,-\gamma}$, one had to ask $-1 \leq \gamma < 0$ even if the electromagnetic field $[E(t,x), B(t,x)]$ enjoys certain temporal decay estimates.

To overcome the last problem, our third idea, which is motivated by the argument developed by T. Hosono and S. Kawashima [38] to deduce the decay property of solutions to nonlinear equations of regularity-loss type, is to design a new time-weighted energy estimate to close the analysis.

Based on the above argument, we can obtain the following results:

- For Cauchy problem of the one-species VPL system (3.13) (1.11), a time-velocity weighted method based on the weight

$$w_{\ell-|\alpha|-|\beta|,\vartheta}(v) = \langle v \rangle^{-(\gamma+2)(\ell-|\alpha|-|\beta|)} e^{q|v|^2/(1+t)^\vartheta}$$

  is introduced in [43], which leads to an extra dissipation term

$$(1+t)^{-1-\vartheta} \left\| \langle v \rangle w_{\ell-|\alpha|-|\beta|,\vartheta}(v) \partial^\alpha_\beta u_2 \right\|^2,$$

  from which one can solve the one-species VPL (3.13) for $-3 \leq \gamma < -2$ without neutral condition but with exponential decay initial perturbation, cf. [43];

- One-species VPB system (1.4):
  - $-2 \leq \gamma < 0$ with neutral condition, cf. [17, 18];
  - $-1 \leq \gamma < 0$ without neutral condition, cf. [66];
- Two-species VMB system (3.4) for $-1 \leq \gamma < 0$, cf. [12];
- Two-species VML system (1.8) for $\gamma \geq -3$, cf. [8, 45, 64].

A natural question is how about the case $-3 < \gamma < -1$ for two-species VMB system (3.4)*? Moreover, for one-species VPB system* (1.6)*, similar analysis requires that $-2 \leq \gamma \leq 1$, then another problem is how about the case $-3 < \gamma < -2$ for one-species VPB system* (1.6)*?*

§ 7. Our second argument

Our second argument is designed to deal with the one-species VPB system (1.6) for the whole range of cutoff potentials. The main ideas are to introduce the weight function $w_{\ell-|\beta|,-\gamma/2}(t,\xi) = \langle \xi \rangle^{-\frac{\gamma}{2}(\ell-|\beta|)} e^{\frac{q\langle \xi \rangle^2}{(1+t)^\vartheta}}$ ($\ell \in \mathbb{R}$, $0 < q \ll 1$), to control the nonlinear terms involving the electric force first and then to close the analysis through one set of energy estimate with respect to the weight function $w_{\ell-|\beta|,-\gamma/2}(t,\xi)$.

In fact, noticing that $w^2_{\ell-|\beta|,-\gamma/2}=w_{\ell-|\beta|,-\gamma/2}w_{\ell-|\beta|-1,-\gamma/2}\langle\xi\rangle^{-\frac{\gamma}{2}}$, the term $\nabla_x\phi\cdot\nabla_\xi u_2$ is no longer a problem since for $0<\alpha_1\leq\alpha$

$$\begin{aligned}&\left(\nabla_x\partial^{\alpha_1}\phi\cdot\nabla_\xi\partial^{\alpha-\alpha_1}_\beta u_2, w^2_{\ell-|\beta|,-\gamma/2}\partial^\alpha_\beta u_2\right)\\ =&\left(\langle\xi\rangle^{-\gamma/2}\cdot\nabla_x\partial^{\alpha_1}\phi\cdot\left(w_{\ell-|\beta|-1,-\gamma/2}\nabla_\xi\partial^{\alpha-\alpha_1}_\beta u_2\right), w_{\ell-|\beta|,-\gamma/2}\partial^\alpha_\beta u_2\right),\end{aligned}$$

which can be bounded by $D^W_{\ell-|\beta|,-\gamma/2}$ defined by (6.3) provided that the electric filed $\nabla_x\phi(t,x)$ decays suitably fast.

But for the linear transport term $\xi\cdot\nabla_x u_2$, due to

$$\left|w_{\ell-|\beta|,-\gamma/2}(t,\xi)\right|^2=w_{\ell-|\beta|,-\gamma/2}(t,\xi)\langle\xi\rangle^{\frac{\gamma}{2}}w_{\ell-|\beta-e_i|,-\gamma/2}(t,\xi),$$

the term involving the mixed spatial and velocity derivatives of such a term can only be controlled as follows

$$\begin{aligned}&\sum_{|\beta_1|=1}\left(\partial_{\beta_1}\xi\cdot\partial^\alpha_{-\beta-\beta_1}\nabla_x u_2, w^2_{\ell-|\beta|,-\gamma/2}\partial^\alpha_\beta u_2\right)\\ &\lesssim\eta\left\|w_{\ell-|\beta|,-\gamma/2}\partial^\alpha_\beta u_2\right\|^2_\nu+C_\eta\left\|w_{\ell-|\beta-e_i|,-\gamma/2}\partial^{\alpha+e_i}_{\beta-e_i}u_2\right\|^2.\end{aligned}\tag{7.1}$$

The problem is how to control the last term in the right hand side of (7.1)*?*

The key point to overcome such a difficulty is based on an observation that, as the order of the $x-$derivatives of the solutions of the one-species VPB system increases, the corresponding temporal decay rate also increase. Based on such an observation, we can define the temporal energy functional $\mathcal{E}_\infty(t)$ as follows

$$\begin{aligned}\mathcal{E}_\infty(t)=\sup_{0\leq\tau\leq t}&\left\{\sum_{(\alpha,\beta)\in U_{\alpha,\beta}}(1+\tau)^{r_{|\alpha|}}\left(\|w_{|\beta|-\ell}\partial^\alpha_\beta u\|^2+\|\nabla^{|\alpha|+1}_x\phi\|^2\right)\right.\\ &+(1+\tau)^{N-\frac12-2(1-p)}\left\|\nabla^{N-2}_x\nabla_\xi u_2\right\|^2\\ &\left.+\sum_{|\alpha|\leq N-2}(1+\tau)^{|\alpha|+\frac32-(1-p)}\left\|\partial^\alpha u_2\right\|^2\right\},\end{aligned}\tag{7.2}$$

where

$$r_{|\alpha|}=\begin{cases}|\alpha|+\frac12, & \text{when } (\alpha,\beta)\in U^{\rm low}_{\alpha,\beta},\\ \frac{N+2|\alpha|}{3}-\frac12, & \text{when } (\alpha,\beta)\in U^{\rm high}_{\alpha,\beta}\end{cases}\tag{7.3}$$

with

$$\begin{aligned}U_{\alpha,\beta}&=\{(\alpha,\beta)\mid|\alpha|+|\beta|\leq N\}=U^{\rm low}_{\alpha,\beta}\cup U^{\rm high}_{\alpha,\beta},\\ U^{\rm low}_{\alpha,\beta}&\equiv\{(\alpha,\beta)\mid|\alpha|+|\beta|\leq N-1\}\cup\{(\alpha,\beta)\mid|\alpha|+|\beta|=N,|\alpha|<N-2\},\\ U^{\rm high}_{\alpha,\beta}&\equiv\{(\alpha,\beta)\mid|\alpha|+|\beta|=N,N-2\leq|\alpha|\leq N\}.\end{aligned}$$

By a dedicate interplay between the above new temporal weight function and the enhanced time decay rate for higher spatial derivatives of the solution, a satisfactory global wellposedness theory is established in [67] for the one-species VPB system (1.6) for the whole range of cutoff potentials.

This argument can not be used to deal with even the two-species VMB system (3.4) since the analysis in [67] relies heavily on the following facts:

- The electric force is a potential force;
- The highest order derivative of the electric potential $\phi(t,x)$ enjoys suitable decay estimate;
- As the order of the derivatives with respect to x increases once, the corresponding temporal decay rate of the solution of the one-species VPB system (1.6) in L^2-norm increases $\frac{1}{2}$.

§ 8. Our third argument

Our third argument is designed to deal with the two-species VMB system (3.4) for the whole range of soft potentials and *the main idea is to introduce two sets of energy estimates:*

(i). The weighted energy method w.r.t. $w_{\ell-|\beta|,-\gamma}(t,v)$ is used to deduce the necessary temporal decay estimates on $[f(t,x,v),E(t,x),B(t,x)]$ togther with the uniform-in-time weighted energy type estimates;

(ii). The weighted energy method w.r.t. $w_{\ell-|\beta|,1}(t,v)$ is used to yield the time increase rates on the weighted energy type estimates on $[f(t,x,v),E(t,x),B(t,x)]$ with the highest order weight w.r.t. v:

- $I^E_{\ell-|\beta|,1}$ and $I^B_{\ell-|\beta|,1}$, the nonlinear terms related to the electromagnetic field, can be bounded by $D^W_{\ell-|\beta|,1}$ if $[E(t,x),B(t,x)]$ decays suitably fast;
- The term related to the linear transport term can be controlled as

$$
\begin{aligned}
I^{lt}_{\ell-|\beta|,1} \lesssim & \eta \left\|w_{\ell-|\beta|,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu \\
&+\left\|w_{\ell-|\beta-e_i|,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2.
\end{aligned}
$$

Here, since $\gamma/2<-\gamma/2-1<1/2$, the last term does not lead to the increase of the order of the weight w.r.t. v.

Our main observation is that the time-velocity weighted energy method works for the two-species VMB system (3.4) for the whole range of cutoff potentials if the two weight functions are suitably chosen such that

- the time increase rates on the weighted energy type estimates on $[f(t,x,v),E(t,x), B(t,x)]$ with the highest order weight w.r.t. v with the corresponding rates independent of the order of the weight;
- $[E(t,x),B(t,x)]$ enjoys some basic temporal decay estimates which are suitably fast.

More precisely, if we define the energy functionals and the corresponding energy dissipation rate functionals as follows:

$$\overline{\mathcal{E}}_{N,\ell,\kappa}(t)\sim\mathcal{E}_{N,\ell,\kappa}(t)+\left\|\Lambda^{-\varrho}(f,E,B)\right\|^2, \tag{8.1}$$

$$\begin{aligned}\overline{\mathcal{D}}_{N,\ell,\kappa}(t)\sim\mathcal{D}_{N,\ell,\kappa}(t)&+\left\|\Lambda^{1-\varrho}(a,b,c,E,B)\right\|^2\\&+\left\|\Lambda^{-\varrho}(a_+-a_-,E)\right\|^2+\left\|\Lambda^{-\varrho}\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2,\end{aligned} \tag{8.2}$$

$$\mathcal{E}_{N,\ell,\kappa}(t)\sim\sum_{|\alpha|+|\beta|\le N}\left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta f\right\|^2+\|(E,B)\|^2_{H^N}, \tag{8.3}$$

$$\begin{aligned}\mathcal{D}_{N,\ell,\kappa}(t)\sim&\sum_{1\le|\alpha|\le N}\|\partial^\alpha(a_\pm,b,c)\|^2+\sum_{|\alpha|+|\beta|\le N}\left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_\nu^2\\&+\|a_+-a_-\|^2+\|E\|^2_{H^{N-1}}+\|\nabla_xB\|^2_{H^{N-2}}\\&+(1+t)^{-1-\vartheta}\sum_{|\alpha|+|\beta|\le N}\left\|\langle v\rangle w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2,\end{aligned} \tag{8.4}$$

$$\mathcal{E}_N(t)\sim\sum_{k=0}^N\left\|\nabla^k(f,E,B)\right\|^2, \tag{8.5}$$

$$\mathcal{E}^k_{N_0}(t)\sim\sum_{|\alpha|=k}^{N_0}\|\partial^\alpha(f,E,B)\|^2, \tag{8.6}$$

$$\mathcal{E}^k_{N_0,\ell,\kappa}(t)\sim\sum_{\substack{|\alpha|+|\beta\le N,\\|\alpha|\ge k}}\left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta f\right\|^2+\sum_{|\alpha|=k}^{N_0}\|\partial^\alpha(E,B)\|^2, \tag{8.7}$$

$$\begin{aligned}\mathcal{D}_N(t)\sim&\|(E,a_+-a_-)\|^2+\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha(\mathbf{P}f,E,B)\|^2\\&+\sum_{|\alpha|=N}\|\partial^\alpha\mathbf{P}f\|^2+\sum_{|\alpha|\le N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu,\end{aligned} \tag{8.8}$$

$$
\begin{aligned}
\mathcal{D}^k_{N_0}(t) \sim & \left\|\nabla^k(E, a_+ - a_-)\right\|^2 + \sum_{k+1\leq|\alpha|\leq N_0-1} \|\partial^\alpha(\mathbf{P}f, E, B)\|^2 \\
& + \sum_{|\alpha|=N_0} \|\partial^\alpha \mathbf{P}f\|^2 + \sum_{k\leq|\alpha|\leq N_0} \|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu,
\end{aligned} \tag{8.9}
$$

$$
\begin{aligned}
\mathcal{D}^k_{N_0,\ell,\kappa}(t) \sim & \left\|\nabla^k(E, a_+ - a_-)\right\|^2 + \sum_{k+1\leq|\alpha|\leq N_0-1} \|\partial^\alpha(\mathbf{P}f, E, B)\|^2 \\
& + \sum_{\substack{|\alpha|+|\beta\leq N_0,\\ |\alpha|\geq k}} \left\|w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu + \sum_{|\alpha|=N_0} \|\partial^\alpha \mathbf{P}f\|^2 \\
& + (1+t)^{-1-\vartheta} \sum_{\substack{|\alpha|+|\beta\leq N_0,\\ |\alpha|\geq k}} \left\|\langle v\rangle w_{\ell-|\beta|,\kappa}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2.
\end{aligned} \tag{8.10}
$$

With the above preparations in hand, under the *a priori assumption* that

$$
\begin{aligned}
X(t) = & \sup_{0\leq s\leq t}\left\{\mathcal{E}_N(s) + \overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(s) + \mathcal{E}_{N-1,l_1,-\gamma}(s)\right\} \\
& + \sup_{0\leq s\leq t}\left\{ \sum_{N_0+1\leq n\leq N} \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq n}} (1+s)^{-\sigma_{n,j}} \left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \right.\\
& + \sum_{N_0+1\leq n\leq N-1} \sum_{|\alpha|=n} \left\|w_{l_1^*,1}\partial^\alpha f\right\|^2 + \sum_{|\alpha|=N} (1+s)^{-\frac{1+\epsilon_0}{2}} \left\|w_{l_1^*,1}\partial^\alpha f\right\|^2 \\
& + \sum_{1\leq n\leq N_0} \sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq n}} (1+s)^{-\sigma_{n,j}} \left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \\
& \left. + \sum_{1\leq n\leq N_0} \sum_{|\alpha|=n} \left\|w_{l_0^*,1}\partial^\alpha f\right\|^2 + \left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \right\} \\
\leq & M
\end{aligned} \tag{8.11}
$$

holds for some sufficiently small $M > 0$ and for some suitably chosen parameters N, N_0, l_0, l^*, l_0^*, l_1^*, l_1 etc., we can deduce the following three types of energy estimates:

- **Estimates of type I under the a priori assumption** (8.11)**:**

$$
\begin{aligned}
& \frac{d}{dt}\overline{\mathcal{E}}_{N_0,l_0+l^*,-\gamma}(t) + \overline{\mathcal{D}}_{N_0,l_0+l^*,-\gamma}(t) \\
& \lesssim \left\|\nabla^2(E,B)\right\|^{\frac{1}{\theta_1}}_{H^{N_0-2}} \widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t) + \sum_{|\alpha|=N_0} \varepsilon\|\partial^\alpha E\|^2,
\end{aligned} \tag{8.12}
$$

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t) \lesssim & \left(\|E\|_{L^\infty_x} + \left\|\nabla^2(E,B)\right\|_{H^{N_0-2}}\right)^{\frac{1}{\theta_2}} \widetilde{\mathcal{D}}_{N,l_1^*,1}(t) \\
& + \mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t),
\end{aligned} \tag{8.13}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{N-1,l_1,-\gamma}(t)+\mathcal{D}_{N-1,l_1,-\gamma}(t)\\
&\lesssim \left\|\nabla^2(E,B)\right\|_{H^{N_0-2}}^{\frac{1}{\theta_3}}\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)+\mathcal{E}_N(t)\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\\
&\quad+\sum_{|\alpha|=N-1}\|\partial^\alpha E\|\left\|\mu^\delta\partial^\alpha f\right\|.
\end{aligned}\tag{8.14}
$$

Here

$$
\begin{aligned}
\widetilde{\mathcal{D}}_{N_0,l_0^*,1}(t)\sim&\sum_{1\le n\le N_0}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}\left\|w_{l_0^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu\\
&+\sum_{1\le n\le N}\sum_{|\alpha|=n}\left\|w_{l_0^*,1}\partial^\alpha f\right\|^2_\nu+\left\|w_{l_0^*,1}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu
\end{aligned}\tag{8.15}
$$

and for $m=N-1$ or N, $\widetilde{\mathcal{D}}_{m,l_1^*,1}(t)$ is defined by

$$
\begin{aligned}
\widetilde{\mathcal{D}}_{m,l_1^*,1}(t)\sim&\sum_{N_0+1\le n\le m}\sum_{\substack{|\alpha|+|\beta|=n,\\|\beta|=j,1\le j\le n}}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu\\
&+\sum_{N_0+1\le n\le m}\sum_{|\alpha|=n}\left\|w_{l_1^*,1}\partial^\alpha f\right\|^2_\nu.
\end{aligned}\tag{8.16}
$$

- **Consequence of the estimates of type I:**
  - Notice that $\theta_i(i=1,2,3)$ can be choose sufficiently small as long as $l_j^*(j=1,2)$ is taken sufficiently large, thus one deduce some nice uniform-in-time estimates based on the above three differential inequalities provided that
    - (i$_1$). The electromagnetic field $[E(t,x),B(t,x)]$ enjoys some temporal decay estimates and $\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\in L^1(\mathbb{R}^+)$;
    - (i$_2$). Some estimates on $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$. For example, if we can not deduce the uniform-in-time bounds on $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, it is suffices to show that the time increase rates of $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$, $\widetilde{\mathcal{D}}_{N-1,l_1^*,1}(t)$, and $\widetilde{\mathcal{D}}_{N,l_1^*,1}(t)$ are finite which are independent of the choices of the parameters l_j^* $(j=0,1)$ but depend only on N and N_0.

- **Estimates of type II under the a priori assumption** (8.11)**:**

$$
\frac{d}{dt}\mathcal{E}^k_{N_0}(t)+\mathcal{D}^k_{N_0}(t)\le 0,\quad k=0,1,\cdots,N_0-2,\tag{8.17}
$$

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}^k_{N_0,\ell,-\gamma}(t)+\mathcal{D}^k_{N_0,\ell,-\gamma}(t)\\
&\lesssim \sum_{|\alpha|=N_0}\|\partial^\alpha E\|^2+\chi_{k\geq 2}\sum_{\substack{1\leq|\alpha'|\leq k-1,\\ |\alpha|+|\beta|=N_0}}\left\|\partial^{\alpha'}(E,B)\right\|^2_{L^\infty_x}\\
&\quad\times\left\|\langle v\rangle^{1-\frac{3\gamma}{2}}w_{\ell-|\beta|-1,-\gamma}\partial^{\alpha-\alpha'}_{\beta+e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|^2, k=0,\cdots,N_0-3.
\end{aligned}\tag{8.18}$$

- **Consequence of the estimate II:**
  - For $0\leq k\leq N_0-2$, the interpolation technique developed in Guo-Wang in [35] implies that

$$\mathcal{E}^k_{N_0}(t)\lesssim \max\left\{\sup_{0\leq\tau\leq t}\overline{\mathcal{E}}_{N_0,N_0+\frac{k+\varrho}{2},-\gamma}(\tau),\sup_{0\leq\tau\leq t}\mathcal{E}_{N_0+k+\varrho}(\tau)\right\}(1+t)^{-(k+\varrho)};$$

  - For $0\leq k\leq N_0-3$, one can further deduce the decay of $\mathcal{E}^k_{N_0,l_0,-\gamma}(t)$ with $\mathcal{E}^1_{N_0,l_0,-\gamma}(t)\in L^1(\mathbb{R}^+)$.

- **Estimates of type III under the a priori assumption** (8.11)**:** The estimates w.r.t. $w_{\ell-|\beta|,1}(t,v)$
  - The linear transport term is a problem

$$I^{lt}_{\ell-|\beta|,1}\lesssim \eta\left\|w_{\ell-|\beta|,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu+\left\|w_{\ell-|\beta-e_i|,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2,$$

    but the last term in the above inequality does not lead to the increase of the order of the weight;
  - Design different time increase rate $\sigma_{n,j}$ for $\sum_{|\alpha|+|\beta|=n,|\beta|=j}\left\|w_{l_1^*-j,1}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2$ with

$$\sigma_{n,j}-\sigma_{n,j-1}=\frac{2(1+\gamma)}{\gamma-2}(1+\vartheta).\tag{8.19}$$

    Thus one can deduce that

$$\begin{aligned}
&\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq n}}(1+t)^{-\sigma_{n,j}}\left\|w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}-1}\right\|^2\\
\lesssim&\sum_{\substack{|\alpha|+|\beta|=n,\\ |\beta|=j,1\leq j\leq N_0}}\left\{(1+t)^{-\sigma_{n,j-1}-1-\vartheta}\left\|w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle\right\|^2\right.\\
&\left.+(1+t)^{-\sigma_{N_0,j-1}}\left\|w_{\ell-j+1,1}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\nu\right\}.
\end{aligned}$$

Based on the above ideas, one can then deduce the global wellposedness of the Cauchy problem of the two-species VMB system (3.4) for the whole range of cutoff potentials, cf. [9].

A by-product of the above argument can be employed to deal with the Cauchy problem of the one-species VPL system (3.14) with algebraic decay perturbation but under neutral condition, cf. [42]. It is worth to pointing out that the neutral condition is imposed to guarantee that the electric potential $\phi(t,x)$ satisfies the decay estimate (5.2), which seems necessary in our analysis.

Before concluding this section, we emphasize that our argument used in this section to treat the two-species VMB system (3.4) relies on the following facts:

- The solution operator of the corresponding linearized system decays sufficiently fast;
- As the order of the derivatives of the solution w.r.t. x increases once, the temporal decay rate for solution operators for both linearized system and nonlinear system in L^2-norm increases $\frac{1}{2}$.

We note, however, that for the one-species VMB system (1.4), the analysis in [7] tells us that

- The dissipation of the magnetic field in one-species case is strictly weaker than the one in two-species case and the solution operator of the corresponding linearized system decays in L^2-norm only like $(1+t)^{-\frac{3}{8}}$;
- As the order of the derivatives of the solution w.r.t. x increases once, the temporal decay rate of the solution operator of the corresponding linearized system in L^2-norm increases only $\frac{1}{4}$ and thus, even for the hard sphere model, one can not deduce the temporal decay estimates on the global solutions by combining the temporal decay estimates on the solution operator of the corresponding linearized system with the Duhamel principle.

Even so, by using the method developed in [35] to establish the optimal time decay rates of the solutions to the Cauchy problem of the compressible Navier-Stokes equations and the Boltzmann equation through the pure energy method and based on a refined time-velocity weighted energy method, partial result for the one-species VMB system (1.4) for $\gamma > -1$ is obtained in [44]. The argument used in [44] can be used to treat the one-species VML system (1.9) to yield a similar global solvability result for $\gamma \geq -3$.

§ 9. Some remarks

In summary, the results concerning the global solvability of the Cauchy problem of the complex kinetic equations mentioned in Section 1 in the perturbative framework can be summarized as follows:

- For the two-species VPL system (1.10), the case with neutral condition imposed on the initial perturbation is given in [56], while the case with general initial perturbation is obtained in [62]. In all these results, the initial perturbations are assumed to be decay algebraically w.r.t. v. For the one-species VPL system (1.11), the case with neutral condition and exponential decay initial perturbation is given in [19], the case with exponential decay initial perturbation is obtained in [43], and the case with algebraically decay initial perturbation but under neutral condition is given in [42]. The problem for the one-species VPL system (1.11) with general algebraically decay initial perturbation is unknown;

- The results for the two-species VML system (1.8) are well-established in [8, 45, 64], while the corresponding result for the one-species VML system (1.9) can also be obtained by employing the argument used in [44] for the one-species VMB system (1.4);

- For the one-species VPB system system (1.6), the case of $-2 \leq \gamma \leq 1$ with neutral condition and exponential decay initial perturbation is given in [17, 18], while the case of $-1 \leq \gamma \leq 1$ for general exponential decay initial perturbation is treated in [66], and a satisfactory global wellposedness theory is established in [67] for the whole range of cutoff potentials and general exponential decay initial perturbation. It would be of some interest to see whether similar result holds for general algebraically decay initial perturbation and all cutoff potentials;

- For the two-species VMB system (3.4), a somewhat satisfactory global solvability result is given in [9] for whole range of cutoff potentials. For the one-species VMB system (1.4), only partial result is obtained in [44]. The problem for the case of $-3 < \gamma \leq -1$ is unknown.

Before concluding this section, we list some related results on the Boltzmann type equations and Landau type equations as in the following:

- The corresponding global solvability results for the Cauchy problem of the Boltzmann equations with non-cutoff potentials are established in [1, 2, 3, 26, 51, 54]. For the corresponding results with external force, cf. [10] for the two-species VPB system (1.5), [65] for the one-species VPB system (1.6), [12] and [20] for the two-species VMB system (3.4). Note that all these results for both VMB system and VPB system are concentrated on the case when the angular singularity is strong, i.e. $\frac{1}{2} \leq s < 1$, the cases with weak angular singularity, i.e. $0 < s < \frac{1}{2}$, are studied in [61] for one-species VPB system (1.6) and in [21] for the two-species VMB system (3.4);

- The spectrum analysis of some kinetic equations, which contain the Landau type equations for $\gamma \geq -2$, Boltzmann type equations with hard potentials for both cutoff and non-cutoff intermolecular interactions, are studied in [24, 23, 39, 46, 47, 69];

- For results on the global solvability in Besov spaces for kinetic equations, see [11, 27, 51, 52, 57] for the Boltzmann equation and [36] for the one-species VPL system (3.14).

Before concluding this section, it is worth to pointing out that, although the main purpose of this paper is to give an updated survey on the perturbation approach to the study on some kinetic equations by energy methods, there is another approach based on semigroup study iniated by S. Ukai in 1970s for the Boltzmann equation, cf. [58, 59].

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